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SOME INTERSECTION THEOREMS

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Let $\mathcal{S}(\mathcal{A})$ be the set of submatrices of an $m \times n$ matrix A. Then $\mathcal{S}(\mathcal{A})$ is a ranked poset with respect to the inclusion, and the poset rank of a submatrix is the sum of the number of rows and columns minus 1, the rank of the empty matrix is zero. We attack the question: What is the maximum number of submatrices such that any two of them have intersection of rank at least t? We have a solution for $t=1,2$ using the followoing theorem of independent interest. Let $m(n,i,j,k)=\max(|\mathcal{F}|+|\mathcal{G}|)$, where maximum is taken for all possible pairs of families of subsets of an n-element set such that $\mathcal{F}$ is i-intersecting, $\mathcal{G}$ is j-intersecting and $\mathcal{F}$ ansd $\mathcal{G}$ are cross-k-intersecting. Then for $i \le j \le k$, $m(n,i,j,k)$ is attained if $\mathcal{F}$ is a maximal i-intersecting family containing subsets of size at least $\frac{n}{2}$, and $\mathcal{G}$ is a maximal $2k-i$-intersecting family.

Furthermore, we discuss an Erdős–Ko–Rado-type question for $\mathcal{S}(\mathcal{A})$, as well.

1. Introduction

In the present paper we continue the work begun in [7], aimed to find generalizations, straightenings or analogons of "classical" extremal theorems. Here we present a generalization of Katona's Theorem and an analogous theorem to the famous Erdős–Ko–Rado Theorem. In the proof of the Katona-type theorem we shall apply a theorem on cross-intersecting families, which is interesting for its own sake. This whole project was initiated by G.O.H. Katona and V.T. Sós. Although this work is a continuation of [7], it is selfcontent.

We shall investigate the following poset. Let A be an $m \times n$ matrix, let M be the set of rows of A, while let N be the set of columns. Let $\mathcal{S}(\mathcal{A})$ denote the collection of submatrices of A. Here a submatrix is considered to be a pair $X=(R,C)$, where R is the set of rows of X and C is the set of columns, i.e. $R \subseteq M$, $C \subseteq N$. X is a submatrix means that either $R \neq \emptyset$ and $C \neq \emptyset$ or $R=C=\emptyset$. Note, that the actual entries of the submatrices are irrelevant, only the positions count. $\mathcal{S}(\mathcal{A})$ is naturally endowed with a poset structure by the inclusion, namely $(R,C) \le (S,D)$ iff $R \subseteq S$ and $C \subseteq D$. With this ordering $\mathcal{S}(\mathcal{A})$ becomes a non-distributive lattice. The intersection and the union of two submatrices can be described easily. Let $X=(R,C)$ and $Y=(S,T)$.

$$X \wedge Y = \begin{cases} (R \cap S, C \cap T) & \text{if } R \cap S \neq \emptyset \text{ and } C \cap T \neq \emptyset \\ (\emptyset,\emptyset) & \text{otherwise} \end{cases}$$

$$X \vee Y = (R \cup S, C \cup T).$$

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$\mathcal{S}(\mathcal{A})$ is the subposet of $2^M \times 2^N$ obtained by dropping all members of the direct product that have exactly one empty component. This makes a significant difference, because there is only one empty submatrix. Furthermore, $\mathcal{S}(\mathcal{A})$ is ranked (or graded) by the following rank function

$$r((R,C)) = \begin{cases} |R|+|C|-1 & \text{if } R \neq \emptyset \\ \emptyset & \text{otherwise.} \end{cases}$$

For a subfamily $\mathcal{F}$ of $\mathcal{S}(\mathcal{A})$ let $\mathcal{F}_N$ and $\mathcal{F}_M$ denote the projections to N and M, respectively. That is, $\mathcal{F}_N = \{C : \exists R\ (R,C) \in \mathcal{F}\}$ and $\mathcal{F}_M = \{R : \exists C\ (R,C) \in \mathcal{F}\}$. For $C \subseteq N$ let $\mathcal{F}[C] = \{R : (R,C) \in \mathcal{F}\}$. Similarly, let $\mathcal{F}\{R\} = \{C : (R,C) \in \mathcal{F}\}$ for $R \subseteq M$. For Sperner properties of $\mathcal{S}(\mathcal{A})$ and a Kruskal-Katona type theorem see [7].

2. The Katona-Type Question

The famous theorem of Katona [4] states that if $\mathcal{F}$ is a collection of subsets of an n-element set, such that any two members of $\mathcal{F}$ intersect in at least t elements, then

$$|\mathcal{F}| \leq \begin{cases} \sum\limits_{i=\frac{n+t}{2}}^{n} \binom{n}{i} & \text{if } n+t \text{ is even} \\ \sum\limits_{i=\frac{n+t+1}{2}}^{n} \binom{n}{i} + \binom{n-1}{\frac{n+t-1}{2}} & \text{if } n+t \text{ is odd} \end{cases}$$

and the above bound is sharp. We shall denote this bound by $K(n,t)$. Here we pose the analogous question for $\mathcal{S}(A)$.

Definition 2.1 We say that a subfamily $\mathcal{F}$ of $\mathcal{S}(A)$ is t-intersecting if for any two members F and G of $\mathcal{F}$ we have that $r(F \wedge G) \geq t$.

The question is how large can be a t-intersecting subfamily of $\mathcal{S}(A)$. The answer for $t=1$ is easy and the main result of this section is the discussion of the case $t=2$.

Proposition 2.2 [7]. *The largest intersecting subfamily $\mathcal{F}$ of $\mathcal{S}(A)$ has size $|\mathcal{F}| = 2^{m+n-2}$.* ∎

For $t=2$ the situation is much more complicated and the maximal family is no longer of direct product type.

Theorem 2.3. *Let $\mathcal{F}$ be a 2-intersecting subfamily of $\mathcal{S}(A)$. Then*

$$|\mathcal{F}| \leq \begin{cases} 2^{n-1}2^{m-1} - \frac{1}{2}\binom{n}{\frac{n}{2}}\binom{m}{\frac{m}{2}} & \text{if } n \text{ and } m \text{ are even} \\ 2^{n-1}2^{m-1} - \frac{1}{2}\binom{n}{\frac{n+1}{2}}\binom{m}{\frac{m}{2}} & \text{if } n \text{ is odd and } m \text{ is even} \\ 2^{n-1}2^{m-1} - \frac{1}{2}\binom{m}{\frac{m+1}{2}}\binom{n}{\frac{n}{2}} & \text{if } m \text{ is odd and } n \text{ is even} \\ 2^{n-1}2^{m-1} - \frac{1}{2}\binom{n}{\frac{n+1}{2}}\binom{m}{\frac{m+1}{2}} & \text{if } n \text{ and } m \text{ are odd.} \end{cases}$$

The bound is sharp in the first three cases and "almost" sharp in the fourth case.

The proof of the theorem is based on the following two lemmae and a theorem of Katona.

Lemma 2.4. *Let $\mathcal{F}$ be a 2-intersecting subfamily of $\mathcal{S}(A)$, C and C' be subsets of N such that $|C \cap C'| = 1$. If none of $\mathcal{F}[C]$ and $\mathcal{F}[C']$ is empty, then $|\mathcal{F}[C]| + |\mathcal{F}[C']| \leq K(m,1) + K(m,3)$.*

Proof. This lemma is an immediate corollary of Theorem 4.3. One just has to observe that $\mathcal{F}$ is 2-intersecting and $|C \cap C'| = 1$ implies that $\mathcal{F}[C]$ and $\mathcal{F}[C']$ are cross-2-intersecting families that are intersecting themselves. ∎

Lemma 2.5. *Let $\mathcal{F}$ be a maximal (with respect to inclusion) 2-intersecting subfamily of $\mathcal{S}(A)$. Then either both $\mathcal{F}_M$ and $\mathcal{F}_N$ are maximal intersecting families of subsets of M and N, respectively, or*

$$|\mathcal{F}| \leq \max\{2^{n-2}(K(m,1) + K(m,3)), 2^{m-2}(K(n,1) + K(n,3))\}.$$

Proof. We prove that if $\mathcal{F}_N$ is not a maximal intersecting family of subsets of N, then $|\mathcal{F}| \leq 2^{m-2}(K(n,1) + K(n,3))$. Similar argument shows that if $\mathcal{F}_M$ is not a maximal intersecting family of subsets of M, then $|\mathcal{F}| \leq 2^{n-2}(K(m,1) + K(m,3))$.

If $\mathcal{F}_N$ is not maximal intersecting, then there is a subset B of N such that neither B nor $N \setminus B$ is in $\mathcal{F}_N$. One of them, say B, could be added to $\mathcal{F}_N$ so that $\mathcal{F}_N$ would remain intersecting. Because (M,B) is not in $\mathcal{F}$, we have a member (U,V) of $\mathcal{F}$, such that $r((M,B) \cap (U,V)) = 1$. That is, $|U| = 1$. Thus every member of $\mathcal{F}_M$ must contain U. Suppose, that $U \subseteq W \subseteq M$ and W is not in $\mathcal{F}_M$. In particular, (W,N) is not in $\mathcal{F}$, wich implies that $\exists (T,S) \in \mathcal{F}$ such that $r((T,S) \cap (W,N)) = 1$. This can happen only if $|S| = 1$, but in this case we have that $r((T,S) \cap (U,V)) = 1$, a contradiction. So if $\mathcal{F}_N$ is not a maximal intersecting family, then $\mathcal{F}_M$ consists of all subsets of M containing a fixed element. Then it is obvious that $\mathcal{F}_M$ can be partitioned into 2^{m-2} pairs of form $\{R, R'\}$ with $|R \cap R'| = 1$. Applying Lemma 2.4 yields the inequality. ∎

Now we state a theorem of Katona, which is usually called the *"Intersecting Kruskal-Katona Theorem"*.

Theorem 2.6 [4]. *Let $1 \leq g \leq l$, $1 \leq k \leq l$ and $g + k \geq l$, further let $\mathcal{A} = \{A_1, A_2, \ldots, A_t\}$ be a k-intersecting system of l-element subsets of an n-element set X. Let $\mathcal{A}^g$ denote the collection of g-element subsets of X contained in at least one of the members of $\mathcal{A}$. Then*

$$t \frac{\binom{2l-k}{g}}{\binom{2l-k}{l}} \leq |\mathcal{A}^g|. \qquad ∎$$

Proof of Theorem 2.3. First, we prove the upper bounds. According to Lemma 2.5, we may assume that both $\mathcal{F}_M$ and $\mathcal{F}_N$ are maximal intersecting families, i.e. they are of size 2^{m-1} and 2^{n-1}, respectively. Let f_i be the number of i-element subsets in $\mathcal{F}_N$ $(i = 0, 1, \ldots, n)$. We distinguish two cases.

Case 1: both n and m are even.

Let $i_0 = \min\{i | f_i > 0\}$. By Theorem 2.6 there exists a matching from the j-element sets in $\mathcal{F}_N$ to the $n - j + 1$-element subsets of N, such that matched pairs intersect in exactly one point, for $i_0 \le j \le \frac{n}{2}$. Indeed, let us denote the j-element sets in $\mathcal{F}_N$ by $A_1, A_2, \ldots, A_{f_j}$. If I is a subset of $\{1, 2, \ldots, f_j\}$, then let ΔI denote the collection of $j-1$-element sets contained in at least one of the A_i's. Now Theorem 2.6 implies that

$$\forall I \subseteq \{1, 2, \ldots, f_j\} \quad |I| \le |\Delta I|$$

From this, applying Kőnig's Theorem follows that there are f_j distinct $j-1$-element sets $B_1, B_2, \ldots, B_{f_j}$ such that $B_i \subset A_i$. Taking $N \backslash B_i$'s we obtain the desired matching. Out of the f_j B_i's at most f_{j-1} are in $\mathcal{F}_N$, so at least $f_j - f_{j-1}$ sets of form $N \backslash B_i$ are in $\mathcal{F}_N$, so at least $f_j - f_{j-1}$ j-element set in $\mathcal{F}_N$ have exactly 1-intersecting $n-j+1$-element pair in $\mathcal{F}_N$. So the number of distinct exactly 1-intersecting pairs in $\mathcal{F}_N$ is at least

$$f_{i_0} + (f_{i_0+1} - f_{i_0}) + \ldots + (f_{\frac{n}{2}} - f_{\frac{n}{2}-1}) = f_{\frac{n}{2}}$$

Because $\mathcal{F}_N$ is maximal intersecting,

$$f_{\frac{n}{2}} = \frac{1}{2}\binom{n}{\frac{n}{2}}$$

So

$$\begin{aligned} |\mathcal{F}| &\le 2 f_{\frac{n}{2}} \frac{K(m,1) + K(m,3)}{2} + (2^{n-1} - 2 f_{\frac{n}{2}}) 2^{m-1} \\ &= \binom{n}{\frac{n}{2}} \frac{K(m,1) + K(m,3)}{2} + (2^{n-1} - \binom{n}{\frac{n}{2}}) 2^{m-1} \\ &= 2^{n-1} 2^{m-1} - \frac{1}{2}\binom{n}{\frac{n}{2}}\binom{m}{\frac{m}{2}} \end{aligned}$$

Case 2: At least one of m and n is odd. We may assume without loss of generality that n is odd. We shall loosen the conditions a little. Instead the the original problem we shall consider the following. Let $\mathcal{H}$ be a maximal intersecting family of subsets of N and let $w : \mathcal{H} \to \mathbb{N}$ be a weight function with the following properties

1) $\forall H \in \mathcal{H} \quad w(H) \le 2^{m-1}$
2) $H, H' \in \mathcal{H} \ |H \cap H'| = 1 \Longrightarrow w(H) + w(H') \le K(m,1) + K(m,3)$

We shall derive an upper bound for the maximum of $\sum_{H \in \mathcal{H}} w(H)$ where the maximum is taken for all possible maximal intersecting families of subsets of N and all possible weightings. Clearly for a 2-intersecting subfamily $\mathcal{F}$ of $\mathcal{S}(A)$, $\mathcal{F}_N$ satisfies 1) and 2) with the weight function $w(C) = |\mathcal{F}[C]|$ and we have

$$|\mathcal{F}| = \sum_{C \in \mathcal{F}_N} w(C)$$

Let h_i denote the number of i-element sets in $\mathcal{H}$ and let $i_0 = \min\{i | h_i > 0\}$. Suppose, that $i_0 \le \frac{n-1}{2}$ and let $\{H_1, H_2, \ldots, H_{h_{i_0}}\}$ be the i_0- element sets in $\mathcal{H}$. Again, applying Theorem 2.6, we obtain $n - i_0 + 1$-element sets $\{G_1, G_2, \ldots, G_{h_{i_0}}\}$ of $\mathcal{H}$ such that $|G_j \cap H_j| = 1$ for $j = 1, 2, \ldots, h_{i_0}$. We change the weight function so that we put $w(H_j) = K(m,3)$ and $w(G_j) = K(m,1)$. In this way the total weight does not decrease and conditions 1) and 2) are still satisfied. Now, if we drop the i_0-element sets from $\mathcal{H}$ and add their complements with weight $K(m,3)$, then the total weight is not changing and conditions 1) and 2) are still satisfied. Repeatedly applying this process, we obtain that $\mathcal{H}$ consists of all subsets of N that are of size at least $\frac{n+1}{2}$. In this case condition 2) has restriction only for sets of size $\frac{n+1}{2}$. Because every $\frac{n+1}{2}$-element subset of N 1-intersects $\frac{n+1}{2}$ others we have that

$$\sum_{\substack{H\in\mathcal{H}\\ |H|=\frac{n+1}{2}}} w(H) \le \binom{n}{\frac{n+1}{2}} \frac{K(m,1)+K(m,3)}{2}$$

So
$$\sum_{H\in\mathcal{H}} w(H) \le \binom{n}{\frac{n+1}{2}} \frac{K(m,1)+K(m,3)}{2} + (2^{n-1} - \binom{n}{\frac{n+1}{2}})2^{m-1}$$

which yields the desired upper bounds for $|\mathcal{F}|$.

We give constructions that will show that the bounds are sharp if at least one of n and m is even. When both n and m are odd, then our construction gives a lower bound for $\max|\mathcal{F}|$ that is very close to the obtained upper bound.

Let $\mathcal{U}$ be a 2-intersecting family of subsets of M of size $K(m,2)$ and $\mathcal{V}$ be a maximal 1-intersecting family of subsets of M containing sets of size at least $\frac{m}{2}$. Note, that $\mathcal{U} \subset \mathcal{V}$. Let both n and m be even and let $\mathcal{F}$ be the subfamily of $\mathcal{S}(A)$ defined as

$$(R,C)\in\mathcal{F} \iff \begin{cases} |C| > \frac{n}{2}+1 & \text{and } R\in\mathcal{V} \\ |C| = \frac{n}{2}+1, & n\notin C \text{ and } R\in\mathcal{V} \\ |C| = \frac{n}{2}+1, & n\in C \text{ and } R\in\mathcal{U} \\ |C| = \frac{n}{2}, & n\notin C \text{ and } R\in\mathcal{U} \end{cases}$$

It is easy to check that $\mathcal{F}$ is 2-intersecting and has the required size. If at least one of n and m is odd, then we may suppose that n is odd. Let in this case $\mathcal{F}$ be defined as

$$(R,C)\in\mathcal{F} \iff \begin{cases} |C| > \frac{n+1}{2} \text{ and } R\in\mathcal{V} \\ |C| = \frac{n+1}{2} \text{ and } R\in\mathcal{U} \end{cases}$$

Using that for even m we have $2K(m,2) = K(m,1)+K(m,3)$ one can check that this construction reaches the upper bound. When m is odd, too, then the above construction gives a 2-intersecting family $\mathcal{F}$ of size $2^{n+m-2} - \frac{m+3}{2m}\binom{n}{\frac{n+1}{2}}\binom{m}{\frac{m+1}{2}}$ instead of $2^{n+m-2} - \frac{1}{2}\binom{n}{\frac{n+1}{2}}\binom{m}{\frac{m+1}{2}}$. ∎

The Erdős–Ko–Rado type problem

The well known Erdős–Ko–Rado Theorem [1] considers t-intersecting families, but in spite of Katona's Theorem here the size of the subsets in the family is fixed. A straightforward analogous condition would be for $\mathcal{S}(\mathcal{A})$ that we require t-intersecting system of rank k submatrices. However, in this form the original Erdős–Ko–Rado Theorem answers the question, because rank k submatrices are special $k+1$-element subsets of $M \cup N$, and they are t-intersecting as matrices iff they are $t+1$-intersecting as sets, with the additional condition that their intersections must have parts both in M and N. Thus, we have more conditions than those in the Erdős–Ko–Rado Theorem, but we have the same lower bound for the size of the maximal family by trivial construction, so the upper bound of the Erdős–Ko–Rado Theorem is sharp for this case, too.

However, if we restrict our attention to $k \times l$ submatrices, then the problem of finding the maximum possible number of t-intersecting $k \times l$ submatrices of A becomes interesting. We can state the following theorem.

Theorem 3.1. *Let $\mathcal{F} \subset \mathcal{S}(\mathcal{A})$ be a collection of t-intersecting $k \times l$ submatrices. If $n > n_0(k,l)$ and $m > m_0(k,l)$, then*

$$|\mathcal{F}| \leq \max\left\{ \binom{m-a}{k-a}\binom{n-(t+1-a)}{l-(t+1-a)} \; : \; 1 \leq a \leq t \right\}.$$

Furthermore, the above bound is sharp.

The proof of this theorem relies heavily on the shifting technique perfected by P. Frankl and Z. Füredi [3]. Also, it involves a theorem of Frankl and the bounds $n_0(k,l)$ and $m_0(k,l)$ are depending on the bounds in that theorem, which are not calculated, so we cannot tell how large m and n must be. First, we extend the definition of shifting to $k \times l$ submatrices. We assume, that $M = \{1,2,\ldots,m\}$ and $N = \{1,2,\ldots,n\}$. Let us note, that by an ingenious application of the cyclic permutation method Pyber [6] found an other generalization of the Erdős–Ko–Rado Theorem.

Definition 3.2. Let $\mathcal{G} \subset \mathcal{S}(\mathcal{A})$ be a collection of $k \times l$ submatrices. Let $G = (R,C) \in \mathcal{G}$ and $i < j$. The ij-leftshift $l_{ij}(G)$ of G is defined as follows:

$$l_{ij}(G) = \begin{cases} (R, C \setminus \{j\} \cup \{i\}) & \text{if } j \in C,\ i \notin C, \text{ and } (R, C \setminus \{j\} \cup \{i\}) \notin \mathcal{G} \\ (R,C) & \text{otherwise.} \end{cases}$$

The ij-upshift $u_{ij}(G)$ is defined similarly:

$$u_{ij}(G) = \begin{cases} (R \setminus \{j\} \cup \{i\}, C) & \text{if } j \in R,\ i \notin R, \text{ and } (R \setminus \{j\} \cup \{i\}, C) \notin \mathcal{G} \\ (R,C) & \text{otherwise.} \end{cases}$$

We define the left- and upshifts of $\mathcal{G}$ by

$$l_{ij}(\mathcal{G}) = \{ l_{ij}(G) \; : \; G \in \mathcal{G} \}$$
$$u_{ij}(\mathcal{G}) = \{ u_{ij}(G) \; : \; G \in \mathcal{G} \}.$$

We list two important properties of shifting.

Proposition 3.3. *Suppose, that $\mathcal{G} \subset \mathcal{S}(\mathcal{A})$ is t-intersecting. Then $l_{ij}(\mathcal{G})$ and $u_{ij}(\mathcal{G})$ are t-intersecting families, as well. Furthermore $|l_{ij}(\mathcal{G})| = |u_{ij}(\mathcal{G})| = |\mathcal{G}|$.* ∎

$\mathcal{G}$ is called *shifted* if for every $i<j$ we have $l_{ij}(\mathcal{G}) = u_{ij}(\mathcal{G}) = \mathcal{G}$.

Proposition 3.4. *For a shifted family $\mathcal{G}$ we have that*

$$\mathcal{G}[C] \subseteq \mathcal{G}[\{1,2,\dots,l\}] \ \forall C \in \mathcal{G}_N \text{and}$$
$$\mathcal{G}\{R\} \subseteq \mathcal{G}\{\{1,2,\dots,k\}\} \ \forall R \in \mathcal{G}_M$$ ∎

Proof of Theorem 3.1.
We may assume by Proposition 3.3 that $\mathcal{F}$ is shifted. Let

$$a = \min\{|R \cap \{1,2,\dots,k\}| \ : \ R \in \mathcal{F}_M\} \text{ and}$$
$$b = \min\{|C \cap \{1,2,\dots,l\}| \ : \ C \in \mathcal{F}_N\}.$$

We distinguish three cases.

Case 1. $a+b \leq t$. Let R_0 and C_0 give the values a and b. Then

$$F_1 = (R_0, \{1,2,\dots,l\} \in \mathcal{F} \text{ and}$$
$$F_2 = (\{1,2,\dots,k\}, C_0) \in \mathcal{F}.$$

However,

$$F_1 \wedge F_2 = (R_0 \cap \{1,2,\dots,k\}, C_0 \cap \{1,2,\dots,l\}).$$

Hence, $r(F_1 \wedge F_2) = a+b-1 \leq t-1$, a contradiction.

Case 2. $a+b \geq t+2$. It is clear that $|\mathcal{F}_M| = O\left(m^{k-a}\right)$ and $|\mathcal{F}_N| = O\left(n^{l-b}\right)$. We may assume without loss of generality that there are numbers a' and b' such that $1 \leq a' < a$, $1 \leq b' \leq b$ and $a'+b' = t+1$. Then we have

$$|\mathcal{F}| \leq |\mathcal{F}_M||\mathcal{F}_N| = O(m^{k-a}n^{l-b}) < \binom{m-a'}{k-a'}\binom{n-b'}{l-b'}.$$

Case 3. $a+b = t+1$. We divide $\mathcal{F}$ into two parts $\mathcal{F} = \mathcal{F}^1 \cup \mathcal{F}^2$ in such way that

$$C_1 \in \mathcal{F}_N^1 \iff |C_1 \cap \{1,2,\dots,l\}| > b \text{ and}$$
$$C_2 \in \mathcal{F}_N^2 \iff |C_2 \cap \{1,2,\dots,l\}| = b.$$

As above, $|\mathcal{F}^1| \leq O(m^{k-a}n^{l-b-1})$. If $\mathcal{F}^2 \neq \emptyset$, then $C_0 = \{1,2,\dots,b,l+1,\dots,2l-b\} \in \mathcal{F}_N^2$. We have for $\forall C \in \mathcal{F}_N^2$ that $\mathcal{F}[C] \subseteq \mathcal{F}[C_0]$, similarly to Proposition 3.4. If $U,V \in \mathcal{F}[C_0]$ satisfy $|U \cap V| \leq a-1$, then $(U, C_0) \in \mathcal{F}$ and $(V, \{1,2,\dots,l\}) \in \mathcal{F}$ while the intersection of these two matrices has rank $\leq t-1$, a contradiction. Hence, $\mathcal{F}[C_0]$ is an a-intersecting family of l-subsets of N. If

$$\left| \bigcap_{R \in \mathcal{F}[C_0]} R \right| \leq a-1,$$

then by Frankl's Theorem [2] $|\mathcal{F}[C_0]| = O(m^{k-a-1})$ and consequently

$$|\mathcal{F}| = \left|\mathcal{F}^1\right| + \left|\mathcal{F}^2\right| \le O(m^{k-a}n^{l-b-1}) + |\mathcal{F}[C_0]||\mathcal{F}_N|$$
$$\le O(m^{k-a}n^{l-b-1}) + O(m^{k-a-1}n^{l-b})$$
$$< \binom{m-a}{k-a}\binom{n-b}{l-b}.$$

If $\left|\bigcap_{R\in\mathcal{F}[C_0]} R\right| > a$, then $\mathcal{F}^2$ is similarly small. So $\left|\bigcap_{R\in\mathcal{F}[C_0]} R\right| = a$. Let $\bigcap_{R\in\mathcal{F}[C_0]} R = A$. If there is a set R_0 in $\mathcal{F}[\{1,2,\ldots,l\}]$ such that $A \not\subset R_0$, then by the usual argument, for every $R \in \mathcal{F}[C_0]$ we have that $(R\setminus A)\cap R_0 \ne \emptyset$, which implies $|\mathcal{F}[C_0]| = O(m^{k-a-1})$ and that $|\mathcal{F}|$ is "small". Thus, we have showed that if $\mathcal{F}$ is not small, then $\exists |A| = a$ such that $\forall R \in \mathcal{F}_M : A \subset R$.

Similar discussion for the other side yields that if $\mathcal{F}$ is large, then $\exists |B| = b$ such that $\forall C \in \mathcal{F}_N : B \subset C$. That is, all members of $\mathcal{F}$ contain the fixed $a \times b$ submatrix (A,B). Clearly, for every $1 \le a \le t$ and $b = t+1-a$ we have such a system $\mathcal{F}_{a,b}$ of cardinality $\binom{m-a}{k-a}\binom{n-b}{l-b}$. This completes the proof. ∎

Remark: The above argument can easily be extended to d-dimensional $n_1 \times n_2 \times \ldots \times n_d$ matrices and families of $k_1 \times k_2 \times \ldots \times k_d$ submatrices.

4. Cross-intersecting families

The solution of the Katona type problem lead to the following definition and problem that we think is interesting for its own sake, too.

Definition 4.1. *Let $|N| = n$ and $\mathcal{F},\mathcal{G} \subseteq 2^N$. We say that $(\mathcal{F},\mathcal{G})$ is an (i,j,k)-*pair *if $\mathcal{F}$ is i-intersecting, $\mathcal{G}$ is j-intersecting and*

$$\forall F \in \mathcal{F} \quad \forall G \in \mathcal{G} \qquad |F \cap G| \ge k,$$

i.e. $\mathcal{F}$ and $\mathcal{G}$ are cross-k-intersecting. Let $m(n,i,j,k) = \max(|\mathcal{F}| + |\mathcal{G}|)$ where the maximum is taken over all (i,j,k)-pairs $(\mathcal{F},\mathcal{G})$ of subfamilies of 2^N.

We suppose that none of i,j and k is zero. There are some immediate properties of $m(n,i,j,k)$.

Proposition 4.2.

$$\text{(i)} \quad i \le i', j \le j' \text{ and } k \le k' \Longrightarrow m(n,i',j',k') \le m(n,i,j,k)$$
$$\text{(ii)} \quad k \le \min\{i,j\} \Longrightarrow m(n,i,j,k) = K(n,i) + K(n,j)$$
$$\text{(iii)} \quad i \le j \le k \Longrightarrow m(n,i,j,k) = m(n,i,k,k).$$

Proof. (i) and (ii) are obvious. For (iii), let $(\mathcal{F},\mathcal{G})$ attain the upper bound $m(n,i,j,k)$. Then $(\mathcal{F}\cup\mathcal{G},\mathcal{F}\cap\mathcal{G})$ is an (i,k,k)-pair that shows $m(n,i,j,k) \le m(n,i,k,k)$. The inequality $m(n,i,j,k) \ge m(n,i,k,k)$ follows from (i). ∎

Because for a k-intersecting family $\mathcal{F}$ $(\mathcal{F},\mathcal{F})$ is an (i,k,k)-pair for $i \le k$, it is tempting to conjecture that $m(n,i,j,k) = 2K(n,k)$ for $1 \le i \le j \le k$. Furthermore, this is true if $i = 1$, $k = 2$ and n is even. However, for $k > 2$ the above equality fails to hold. Instead, we have the following theorem.

Theorem 4.3. *Let $1 \le i \le j \le k$. Then*

$$m(n,i,j,k) = \begin{cases} \sum_{r=\frac{n+i}{2}}^{n} \binom{n}{r} + \sum_{r=\frac{n+2k-i}{2}}^{n} \binom{n}{r} & \text{if } n+i \text{ is even} \\ \sum_{r=\frac{n+i+1}{2}}^{n} \binom{n}{r} + \sum_{r=\frac{n+2k-i+1}{2}}^{n} \binom{n}{r} + \binom{n-1}{\frac{n+i-1}{2}} + \binom{n-1}{\frac{n+2k-i-1}{2}} & \text{if } n+i \text{ is odd.} \end{cases}$$

Proof. First note, that the above expression for $m(n,i,j,k)$ is same as $m(n,i,j,k) = K(n,i) + K(n,2k-i)$. Taking $\mathcal{F}$ to be a maximal i-intersecting family, $\mathcal{G}$ to be a maximal $2k-i$-intersecting family such that $\mathcal{G} \subseteq \mathcal{F}$, we obtain that $m(n,i,j,k) \ge K(n,i) + K(n,2k-i)$. We need only to prove the inequality in the other direction.

It is enough to consider the case when $\mathcal{F}$ is i-intersecting and $\mathcal{G}$ is k-intersecting by Proposition 4.2. Let $f = \min\{|F| : F \in \mathcal{F}\}$ and $g = \min\{|G| : G \in \mathcal{G}\}$. We distinguish several cases. In each of the cases we shall change $\mathcal{F}$ and $\mathcal{G}$ preserving $|\mathcal{F}| + |\mathcal{G}|$ or show that $|\mathcal{F}| + |\mathcal{G}| \le K(n,i) + K(n,2k-i)$.

Case 1. $f > g-k+i$. Here we apply Theorem 2.6 for the g-element sets of $\mathcal{G}$ to obtain a matching of them into $n-g+k-1$-element sets in such way that matched pairs are exactly $k-1$-intersecting. These $n-g+k-1$-element sets therefore cannot be in either $\mathcal{F}$ or $\mathcal{G}$.

Case 1a. $\frac{n+2k-i-2}{2} \ge g$. Let us drop the g-element sets from $\mathcal{G}$ and add their matched pairs of $n-g+k-1$-element sets to $\mathcal{F}$. Because $f+n-g+k-1 > n+i-1$, all the new sets in $\mathcal{F}$ i-intersect the old ones. Two new sets in $\mathcal{F}$ are i-intersecting because $2(n-g+k-1) \ge n+i$. Finally, because the smallest sets in the new $\mathcal{G}$ have at least $g+1$ elements, the new sets in $\mathcal{F}$ k-intersect all members of the new $\mathcal{G}$, too.

Case 1b. $\frac{n+2k-i-1}{2} \le g$. In this case $f > \frac{n+2k-i-1}{2} - k + i = \frac{n+i-1}{2}$. Hence, $\mathcal{F}$ and $\mathcal{G}$ consist of large sets that implies $|\mathcal{F}| + |\mathcal{G}| \le K(n,i) + K(n,2k-i)$.

Case 2. $f < g-k+i$. We apply Theorem 2.6 for the f-element sets in $\mathcal{F}$ to obtain a matching of them into $n-f+i-1$-element subsets, such that matching pairs are exactly $i-1$-intersecting.

Case 2a. $f < \frac{n+i-1}{2}$. We drop the f-element sets from $\mathcal{F}$ and add their matched $n-f+i-1$-element pairs to $\mathcal{F}$. Clearly, $\mathcal{F}$ remains i-intersecting. Furthermore, $g+n-f+i-1 > f+k-i+n-f+i-1 = n+k-1$ implies that the new elements of $\mathcal{F}$, too, k-intersect all members of $\mathcal{G}$.

Case 2b. $f \ge \frac{n+i-1}{2}$. Here we have that $g > f+k-i = \frac{n+2k-i-1}{2}$. Hence, $|\mathcal{F}| + |\mathcal{G}| \le K(n,i) + K(n,2k-i)$.

Case 3. $f = g-k+i$.

Case 3a. $f < \frac{n+i-1}{2}$. In this case we match both the f-element sets in $\mathcal{F}$ and the g-element sets in $\mathcal{G}$ to $n-f+i-1$-element and $n-g+k-1$-element subsets, respectively. Then we drop the f- and g-element sets and add their matched pairs to $\mathcal{F}$ and $\mathcal{G}$, respectively. Clearly $\mathcal{F}$ remains i-intersecting and $\mathcal{G}$ remains k-intersecting. Furthermore, $n-f+i-1+g+1 \ge n+k$ and $n-g+k-1+f-1 \ge n+k$ implies that $\mathcal{F}$ and $\mathcal{G}$ remain cross-k-intersecting.

Case 3b. $f \geq \frac{n+i}{2}$. In this case we have that $|\mathcal{F}| \leq K(n,i)$ and $|\mathcal{G}| \leq K(n,2k-i)$ holds trivially.

Case 3c. $f = \frac{n+i-1}{2}$. Then we have that $g = \frac{n+2k-i-1}{2}$. Let a denote the number g element sets in $\mathcal{G}$ and b denote that of the f-element sets in $\mathcal{F}$. We want to prove that

$$a + b \leq \binom{n-1}{\frac{n+i-1}{2}} + \binom{n-1}{\frac{n+2k-i-1}{2}}. \tag{4.1}$$

Let $\delta_{\frac{n-i+1}{2}} a$ denote the number of $\frac{n-i+1}{2}$-element subsets of N that are contained in some of the g-element sets of $\mathcal{G}$. If H is such an $\frac{n-i+1}{2}$-element subset contained in G $(G \in \mathcal{G}, |G| = g)$, then $|N \setminus H| = \frac{n+i-1}{2}$ and $|(N \setminus H) \cap G| = k-1$, so $N \setminus H$ cannot be in $\mathcal{F}$. Hence,

$$a + b \leq a + \binom{n-1}{\frac{n+i-1}{2}} - \delta_{\frac{n-i+1}{2}} a. \tag{4.2}$$

Thus, to prove (4.1) it is enough to show that

$$a - \binom{n-1}{\frac{n+2k-i-1}{2}} \leq \delta_{\frac{n-i+1}{2}} a - \binom{n-1}{\frac{n+i-3}{2}}. \tag{4.3}$$

We use Lovász' [5] version of the Kruskal-Katona Theorem. Let

$$a = \binom{n-1}{\frac{n+2k-i-1}{2}} + \binom{x}{\frac{n+2k-i-3}{2}}. \tag{4.4}$$

(We may assume that $a > \binom{n-1}{\frac{n+2k-i-1}{2}}$ because otherwise (4.1) follows from the fact that there can be at most $\binom{n-1}{\frac{n+i-1}{2}}$ $\frac{n+i-1}{2}$-element sets in an i-intersecting family by the Erdős–Ko–Rado Theorem.) Now Lovász' result tells us that

$$\delta_{\frac{n-i+1}{2}} a \geq \binom{n-1}{\frac{n-i+1}{2}} + \binom{x}{\frac{n-i-1}{2}}. \tag{4.5}$$

By (4.5) (4.3) reduces to

$$\binom{x}{\frac{n+2k-i-3}{2}} \leq \binom{x}{\frac{n-i-1}{2}} \qquad \text{where } x \leq n-2. \tag{4.6}$$

The inequality (4.6) follows from an easy induction argument on k. Because whenever we changed $\mathcal{F}$ or $\mathcal{G}$ the sum $f+g$ was increased, the changing process must terminate. This completes the proof of Theorem 4.3. ∎

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